Is $(\mathbb{R}, +)$ group isomorphic to $(\mathbb{R}^2, +)$?

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Introduction

In this short document we will show how $(\mathbb{R}, +)$ and $(\mathbb{R}^2, +)$ are isomorphic as groups. While researching this problem I found that not many places online have the proof in a rigorous enough way for my liking. Here we will give the proof in its entirety. The only prerequesites are elementary ideas on linear algebra, cardinality, and cardinal multiplication.

The Proof

Theorem. *As additive groups, $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic.*

To prove this, we will show that $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic as $\mathbb{Q}$-vector spaces, which must mean that they're isomorphic as additive groups. We need two preliminary results to continue.

Lemma 1 (Schrödinger-Bernstein)**.** *Let A and B be sets. If there exists injective functions $f : A \to B$ and $g : B \to A$, then there exists a bijective function $h : A \to B$.*

Proof. First, construct sequences $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ such that $A_0 = A$, $B_0 = B$, and for $n > 0$, $B_{n+1} = (A_n)f$ and $A_{n+1} = A \setminus (B \setminus B_{n+1})g$.

We will show that for all $n \in \mathbb{N}$ that $B_{n+1} \subseteq B_n$. For $n = 0$, $B_1 = (A)f \subseteq B = B_0$. Assume that this inclusion is true for some $n = k$. Then $B_{k+1} = (A_k)f = (A \setminus (B \setminus B_k)g)f$. By hypothesis, $B_{k+1} \subseteq B_k$, so $(B \setminus B_{k+1})g \supseteq (B \setminus B_k)g \Rightarrow A_{k+1} = A \setminus (B \setminus B_{k+1})g \subseteq A \setminus (B \setminus B_k)g = A_k \Rightarrow (A_{k+1})f \subseteq (A_k)f \Rightarrow B_{k+2} \subseteq B_{k+1}$. By induction, $\{B_n\}_{n\in\mathbb{N}}$ is a decreasing sequence. A similar argument can be made to show that $\{A_n\}_{n\in\mathbb{N}}$ is a decreasing sequence.

Let $\bar{A} = \bigcap_{n\in\mathbb{N}} A_n$ and $\bar{B} = \bigcap_{n\in\mathbb{N}} B_n$. We will show that $f|_{\bar{A}}$ is a bijection between $\bar{A}$ and $\bar{B}$ and that $g|_{B\setminus\bar{B}}$ is a bijection between $B \setminus \bar{B}$ and $A \setminus \bar{A}$. Since both of these functions inherit injectivity from their unrestricted versions, surjectivity is the only condition to prove.

Let $a \in \bar{A}$. This means for all $n \in \mathbb{N}$, $a \in A_n$ and hence $(a)f \in B_{n+1} \Rightarrow (a)f \in \bar{B}$. Since $a \in \bar{A}$ was arbitrary, we've shown that $(\bar{A})f \subseteq \bar{B}$. Conversely, let $b \in \bar{B}$. For all $n \in \mathbb{N}$, $b \in B_n$ which means for each n there exists an $a_n \in A_n$ such that $(a_n)f = b$. By injectivity, all of these a_n's are equal. Since b was arbitrary, we've shown that for any $b \in \bar{B}$ there exists an $a \in \bar{A}$ such that $(a)f = b$ and therefore $\bar{B} \subseteq (\bar{A})f$.

Now, let $b \in B \setminus \bar{B}$. This means there exists a $n \in \mathbb{N}$ such that $b \notin B_{n+1}$, so as $A_{n+1} = A \setminus (B \setminus B_{n+1})g$, $(b)g \notin A_{n+1}$, and so $(b)g \in A \setminus \bar{A}$. As b was arbitrary, $(B \setminus \bar{B})g \subseteq A \setminus \bar{A}$. Conversely, let $a \in A \setminus \bar{A}$. This means there exists a $n \in \mathbb{N}$ such that $a \notin A_{n+1}$, so $a \in (B \setminus B_{n+1})g$, and hence there exists a $b \in B \setminus B_{n+1}$ such that $(b)g = a$ which will be unique as g is injective. We've shown that for any $a \in A \setminus \bar{A}$ there exists a $b \in B \setminus \bar{B}$ such that $(b)g = a$, therefore $A \setminus \bar{A} \subseteq (B \setminus \bar{B})g$.

Now the function $h : A \to B$ defined by

$$(a)h = \begin{cases} (a)f & \text{if } a \in \bar{A}, \\ (a)g^{-1} & \text{if } a \in A \setminus \bar{A}. \end{cases}$$

is a bijection. □

Lemma 2. *Let X be an infinite set and let S be the set of all finite subsets of X. Then $|X| = |S|$.*

Proof. First, $|X| = |\{\{x\}|x \in X\}| \leq |S|$ since singletons are finite subsets of X. Let $S_n = \{s \in S||s| = n\}$. Thinking of ordered pairs, we know that $|X|^n = |X|$, and since any element in S_n can be ordered $n!$ different ways to create an ordered pair of length n, we get $|S_n| \leq n!|X|^n$. So

$$|X| \leq |S| = \sum_{n\in\mathbb{N}} |S_n| \tag{1}$$

$$\leq \sum_{n\in\mathbb{N}} |X| \tag{2}$$

$$= |\mathbb{N}||X| = |X| \text{ (cardinal multiplication)}. \tag{3}$$

Therefore $|X| = |S|$. □

The next result will give us a direction to pursue for the rest of the proof.

Lemma 3. *Let V and W be F-vector spaces for some field F. If $dimV = dimW$, then $(V, +) \cong (W, +)$.*

Proof. Let $B_V = \{e_i\}_{i\in I}$ and $B_W = \{f_j\}_{j\in J}$ be bases for V and W respectively for some index sets I and J. Since $\dim V = \dim W$, there exists a bijection $\varphi : B_V \to B_W$. We can extend φ to $\bar{\varphi} : V \to W$ like so: given $v = \sum_{i\in I} a_i e_i$ $(a_i \in F)$, we map this to $v\bar{\varphi} = \sum_{i\in I} a_i(e_i)\varphi$. We will show that this function is a linear isomorphism.

First, let $w = \sum_{j\in J} a_j f_j \in W$, then the vector $\sum_{j\in J} a_j(f_j)\varphi^{-1} \in V$ maps to w. So $\bar{\varphi}$ is a surjection.

Next, suppose $v_1, v_2 \in V$ with $v_1 = \sum_{i\in I} a_i e_i$ and $v_2 = \sum_{i\in I} b_i e_i$. If $v_1\bar{\varphi} = v_2\bar{\varphi}$, then $\sum_{i\in I}(a_i - b_i)(e_i)\varphi = 0$ and hence, since B_V is linearly independent, $a_i = b_i$ for all $i \in I$. This proves that $\bar{\varphi}$ is injective. We've shown that $\bar{\varphi}$ is a bijection, and since it's linear by construction, is a linear isomorphism. Therefore $(V, +) \cong (W, +)$. □

So, to show that $(\mathbb{R}, +)$ and $(\mathbb{R}^2, +)$ are isomorphic we need to show that $\dim\mathbb{R} = \dim\mathbb{R}^2$ with respect to some field. This next result hints that we should use a field that will make $\mathbb{R}$ and $\mathbb{R}^2$ infinite dimensional.

Lemma 4. *If V is an infinite dimensional F-vector space for some field F with basis $B_V = \{e_i\}_{i\in I}$, then $|V| = max(|B_V|, |F|)$.*

Proof. Let $v = \sum_{i\in I} a_i e_i \in V$ and define $\delta_v : B_v \to F$ by $e_i\delta_v = a_i$. Let Δ be the set of all possible functions from B_V to F where only finitely many images of elements of B_V are nonzero. Let $\sigma_1 : V \to \Delta$ be defined by $v\sigma_1 = \delta_v$. This is an injection since every element of a vector space can be represented uniquely as a linear combination of its basis. Let $\sigma_2 : \Delta \to V$ be defined by $\pi\sigma_2 = \sum_{e\in B_V} e(e\pi)$, which is an injection for the same reason. By Lemma 1, $|V| = |\Delta|$.

We now want to count the number of functions in Δ. We can represent a function $\delta \in \Delta$ by a finite subset H of $B_V \times F$: $(v, a) \in H$ if and only if $v\delta = a \in F\backslash\{0\}$. Using this, we can get the cardinality of Δ. First, Δ has cardinality at least $|\{\{(v, a)\}|v \in B_V, a \in F\}| = |B_V \times F|$, since each singleton $\{(v, a)\}$ corresponds to the vector av. If S is the set of all finite subsets of $B_V \times F$, then using Lemma 2 we get that $|B_V \times F| = |S|$. However, we've shown that every function in Δ can be represented by a finite subset of $B_V \times F$. Hence

$$|\Delta| \leq |S| = |B_V \times F| \leq |\Delta|$$

and therefore $|\Delta| = |B_V \times F| = |B_V||F| = \max(|B_V|, |F|)$ by the definition of cardinal multiplication.

□

This tells us that we want to find a field F such that $\mathbb{R}$ and $\mathbb{R}^2$ as F-vector spaces have bases which have a larger cardinality than F. We'll then use Lemma 1 to show that that they have equal cardinality, and hence by the above, dimension. The following result tells us what choice for F we should make.

Lemma 5. *Let V be a finite dimensional $\mathbb{Q}$-vector. Then V is countable.*

Proof. Any vector in V can be represented uniquely by an ordered pair from $\mathbb{Q}^{\dim V}$. Since the Cartesian product of countable sets is countable, V must be countable. $\square$

In particular, what the above lemma tells us is that $\mathbb{R}$ and $\mathbb{R}^2$ over $\mathbb{Q}$ are infinite dimensional. We only need one more result to complete the puzzle.

Lemma 6. $|\mathbb{R}| = |\mathbb{R}^2|$.

Proof. We will do this using Lemma 1. First, note that the result is equivalent to showing there exists a bijection $h : (0,1) \to (0,1)^2$ since $h' : (0,1) \to \mathbb{R}$ defined by $xh' = \tan \pi(x - \frac{1}{2})$ is a bijection. Let $f : (0,1) \to (0,1)^2$ be defined by $xf = (x, x)$. This is an injection. For the reverse injection, the story is a bit more intricate.

If $(0.a_1a_2a_3\ldots, 0.b_1b_2b_3\ldots) \in (0,1)^2$, then a natural choice for its image in $(0,1)$ would be $0.a_1b_1a_2b_2a_3b_3\ldots$, but binary representation of numbers are not unique; take for example 0.5 and $0.4999\ldots$. However, since we are looking to construct an injection and not a bijection, we can get around this problem. Let $g : (0,1)^2 \to (0,1)$ be defined by $(0.a_1a_2a_3\ldots, 0.b_1b_2b_3\ldots)g = 0.a_1b_1a_2b_2a_3b_3\ldots$, and if a number has more than one binary representation, pick one arbitrarily to use. This is an injection.

Hence by Lemma 1, there is a bijection between $(0,1)$ and $(0,1)^2$ and therefore a bijection between $\mathbb{R}$ and $\mathbb{R}^2$. $\square$

We now have everything we need to prove our objective. All we need to do is put the pieces together.

Theorem. *As additive groups, $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic.*

Proof. Let $B_{\mathbb{R}}$ and $B_{\mathbb{R}^2}$ be bases for $\mathbb{R}$ and $\mathbb{R}^2$ over $\mathbb{Q}$, respectively. By Lemma 5, we know that $|B_{\mathbb{R}}|$ and $|B_{\mathbb{R}^2}|$ are infinite. Since $|\mathbb{Q}| = |\mathbb{N}|$, the smallest cardinal, we know that $|B_{\mathbb{R}}| \geq |\mathbb{Q}|$ and $|B_{\mathbb{R}^2}| \geq |\mathbb{Q}|$. Hence by Lemma 4, $|\mathbb{R}| = \max(|B_{\mathbb{R}}|, |\mathbb{Q}|) = |B_{\mathbb{R}}|$, and similarily, $|\mathbb{R}^2| = |B_{\mathbb{R}^2}|$. Using Lemma 6, $|B_{\mathbb{R}}| = |B_{\mathbb{R}^2}|$, and finally by Lemma 3, $(\mathbb{R}, +) \cong (\mathbb{R}^2, +)$. $\square$